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Relativistic Theory of Nuclear Forces

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A theory of nuclear forces is proposed which results in a nonlinear relativistic equation for their potential. This equation is used to explain nuclear saturation.

Let us consider the movement of a particle having mass m in a nuclear field that is described by the scalar potential φ. It would be rational to obtain the equation of this movement in an inertial frame of reference by means of the differential law of the conservation of energy and momentum:

$$\partial T^{ik}/\partial x^i = 0 \tag{1}$$

where the energy-momentum tensor T^{ik} has the form (Bogolubov and Shirkov, 1984)

$$T^{ik} = c^2 \rho \frac{dx^i}{ds}\frac{dx^k}{ds} + \frac{1}{\lambda}\left[\left(g^{in}g^{kl} - \frac{1}{2}g^{ik}g^{nl}\right)\frac{\partial\varphi}{\partial x^n}\frac{\partial\varphi}{\partial x^l} + g^{ik}\frac{m_\pi^2 c^2}{2\hbar^2}\varphi^2\right] \tag{2}$$

where ρ is the density of the particle mass at rest, λ is a constant, m_π is the mass of the neutral pion at rest, as it is the carrier of the nuclear interaction which preserves the charges of the interacting particles, $ds^2 = g_{ik}\, dx^i\, dx^k$, where g_{ik} is the Minkowski tensor, and the potential φ is described by the equation

$$\frac{\partial^2\varphi}{\partial x^n\,\partial x_n} + \frac{m_\pi^2 c^2}{\hbar^2}\varphi = -\frac{\mu\rho}{m_p} \tag{3}$$

where μ is a constant and m_p is the proton mass at rest.

This approach to the problem is unacceptable because of the following reasons:

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1. As will be shown later, the system of equations (1)–(3) results in the following contradictory equation of the particle's movement:

$$\frac{d^2x^k}{ds^2} = \gamma g^{ik}\frac{\partial\varphi}{\partial x^i}, \qquad \gamma = \frac{\mu}{\lambda m_p c^2} \tag{4}$$

The contradiction is evident from the following considerations. If we multiply equation (4) by $g_{lk}\, dx^l/ds$ and take into account that ds is the Minkowski interval, then we obtain the equality

$$\frac{1}{2}\frac{d}{ds}\left(g_{lk}\frac{dx^l}{ds}\frac{dx^k}{ds}\right) = \gamma\frac{d\varphi}{ds}$$

Since the left-hand side of this equality is zero, we get the following result: $d\varphi/ds = 0$, which, as a rule, cannot take place.

2. The linear equation (3) can give the right description of the nuclear potential only for relatively small values of the potential φ and its gradient. When the distance between interacting particles is small or the value of φ is large, then equation (3) contradicts the experimental data. Nuclear forces can both attract and repulse, contrary to (3), which results in the well-known phenomenon of nuclear saturation (Naumov, 1984).

To overcome these difficulties, let us assume that the particle mass m depends on the nuclear potential φ. On the basis of this assumption we represent the density ρ in (2) as follows:

$$\rho = \rho_0 f(\varphi), \qquad f(0) = 1 \tag{5}$$

where ρ_0 is the density of the particle mass at rest when $\varphi = 0$ and $f(\varphi)$ is an unknown function.

Then from (1) and (2) we obtain

$$c^2\frac{\partial}{\partial x^i}\left(\rho_0 f\frac{dx^i}{ds}\right)\frac{dx^k}{ds} + c^2\rho\frac{\partial}{\partial x^i}\left(\frac{dx^k}{ds}\right)\frac{dx^i}{ds}$$
$$+\frac{1}{\lambda}\left[\left(g^{in}g^{kl} - \frac{1}{2}g^{ik}g^{nl}\right)\frac{\partial}{\partial x^i}\left(\frac{\partial\varphi}{\partial x^n}\frac{\partial\varphi}{\partial x^l}\right) + g^{ik}\frac{m_\pi^2 c^2}{\hbar^2}\varphi\frac{\partial\varphi}{\partial x^i}\right] = 0 \tag{6}$$

Equation (6) can be represented as follows:

$$\left[f\frac{\partial}{\partial x^i}\left(\rho_0\frac{\partial x^i}{\partial s}\right) + \rho_0\frac{\partial f}{\partial x^i}\frac{dx^i}{ds}\right]\frac{dx^k}{ds} + \rho\frac{d^2x^k}{ds^2}$$
$$+\frac{1}{\lambda c^2}g^{ik}\frac{\partial\varphi}{\partial x^i}\left(\frac{\partial^2\varphi}{\partial x^n\,\partial x_n} + \frac{m_\pi^2 c^2}{\hbar^2}\varphi\right) = 0 \tag{7}$$

Let us assume that m_0 denotes the mass at rest of the particle when $\varphi = 0$. Then the differential equation expressing the conservation of this mass has the well-known form (Landau and Lifshitz, 1973)

$$\frac{\partial}{\partial x^i}\left(\rho_0 \frac{dx^i}{ds}\right) = 0 \tag{8}$$

Therefore, formula (7) acquires the form

$$\rho\left(\frac{d^2x^k}{ds^2} + \frac{1}{f}\frac{df}{ds}\frac{dx^k}{ds}\right) + \frac{1}{\lambda c^2} g^{ik} \frac{\partial \varphi}{\partial x^i}\left(\frac{\partial^2 \varphi}{\partial x^n \partial x_n} + \frac{m_\pi^2 c^2}{\hbar^2}\varphi\right) = 0 \tag{9}$$

After multiplying equation (9) by $g_{lk}\, dx^l/ds$ we obtain

$$\frac{\rho}{2}\frac{d}{ds}\left(g_{lk}\frac{dx^l}{ds}\frac{dx^k}{ds}\right) + \frac{\rho}{f}\frac{df}{ds} + \frac{1}{\lambda c^2}\frac{d\varphi}{ds}\left(\frac{\partial^2 \varphi}{\partial x^n \partial x_n} + \frac{m_\pi^2 c^2}{\hbar^2}\varphi\right) = 0 \tag{10}$$

Since the first summand in (10) is zero, this equation gives

$$\frac{\partial^2 \varphi}{\partial x^n \partial x_n} + \frac{m_\pi^2 c^2}{\hbar^2}\varphi = -\lambda c^2 \rho \frac{1}{f}\frac{df}{d\varphi} \tag{11}$$

Therefore, from (9) and (11) we derive

$$\frac{d^2x^k}{ds^2} + \frac{d(\ln f)}{ds}\frac{dx^k}{ds} - g^{ik}\frac{\partial(\ln f)}{\partial x^i} = 0 \tag{12}$$

Equations (10) and (11) have no contradictions only when the following derivative is not equal to zero: $df/d\varphi \neq 0$. Equation (12) is not contradictory either, as after multiplying it by $g_{lk}\, dx^l/ds$ we get an identity. This identity shows that the first equation in (12) is derived from the other three.

Let us consider the nonrelativistic case. Then equation (12) acquires the form

$$\frac{d^2x^\alpha}{dt^2} = -\frac{\partial(c^2 \ln f)}{\partial x^\alpha}, \qquad \alpha = 1, 2, 3, \qquad t = \frac{x^0}{c} \tag{13}$$

It follows from (13) that the function $c^2 \ln f$ can be identified with the potential φ:

$$c^2 \ln f = \varphi \tag{14}$$

From (14) we find the function $f(\varphi)$ which also satisfies condition (5): $f(0) = 1$, and

$$f(\varphi) = \exp(\varphi/c^2) \tag{15}$$

Thus equation (13) takes the form

$$\frac{d^2x^\alpha}{dt^2} = -\frac{\partial \varphi}{\partial x^\alpha} \tag{16}$$

From (5) and (15) we derive

$$\rho = \rho_0 \exp(\varphi/c^2) \tag{17}$$

After substituting (15) for $f(\varphi)$ in (11) and (12) we obtain

$$c^2 \frac{d^2x^k}{ds^2} + \frac{d\varphi}{ds}\frac{dx^k}{ds} - \frac{\partial \varphi}{\partial x_k} = 0 \tag{18}$$

$$\frac{\partial^2 \varphi}{\partial x^n \partial x_n} + \frac{m_\pi^2 c^2}{\hbar^2}\varphi = -\lambda\rho, \qquad \rho = \rho_0 \exp\left(\frac{\varphi}{c^2}\right) \tag{19}$$

Instead of λ we can use the standard strong interaction constant G. Then we have

$$\lambda = 4\pi G^2/m_p^2 \tag{20}$$

To explain this, let us consider equations (18) and (19) in the nonrelativistic case of the interaction of two protons, whose masses are m_p, when $|\varphi| \ll c^2$. Then in this case, after substituting (20) for λ in (19), we obtain the well-known equations in which the value $m_p\varphi$ is taken as the nuclear potential.

After making the transition from the differential law of energy-momentum conservation (1) to the integral law of energy-momentum conservation and taking (2) and (17) into account, we obtain the following result. The relativistic energy E and mass m of a particle can be determined by

$$E = mc^2, \qquad m = m_0 \exp(\varphi/c^2)/(1 - v^2/c^2)^{1/2} \tag{21}$$

where v is the speed of the particle and m_0 is its mass at rest when $\varphi = 0$.

After the use of formula (21), equation (18) takes the form

$$\frac{d}{dt}\left(m\frac{dx^\alpha}{dt}\right) = -\left(1 - \frac{v^2}{c^2}\right)m\frac{\partial\varphi}{\partial x^\alpha}, \qquad \alpha = 1, 2, 3 \tag{22}$$

$$\frac{d(\ln E)}{dt} = \left(1 - \frac{v^2}{c^2}\right)\frac{\partial\varphi}{c^2\partial t}, \qquad t = \frac{x^0}{c} \tag{23}$$

Let us consider equation (23) in the stationary case: $\partial\varphi/\partial t = 0$. Then from (23) we get

$$mc^2 = m_0 c^2 \exp(\varphi/c^2)/(1 - v^2/c^2)^{1/2} = \text{const} \tag{24}$$

In the nonrelativistic case, after taking the logarithm of expression (24), we obtain

$$m_0 v^2/2 + m_0\varphi = \text{const} \tag{25}$$

Formula (25) represents the classical nonrelativistic law of energy conservation where the summand $m_0\varphi$ is the potential energy.

If we also have an electromagnetic field having potentials A_i, then, instead of (9), the differential law of energy-momentum conservation (1) results in the following equation of the particle movement:

$$\rho\left(c^2\frac{d^2x^k}{ds^2}+\frac{d\varphi}{ds}\frac{dx^k}{ds}\right)+\frac{1}{\lambda c^2}g^{ik}\frac{\partial\varphi}{\partial x^i}\left(\frac{\partial^2\varphi}{\partial x^n\partial x_n}+\frac{m_\pi^2c^2}{\hbar^2}\varphi\right)$$
$$-\theta_0 g^{ik}F_{in}\frac{dx^n}{ds}=0,\qquad F_{in}=\frac{\partial A_n}{\partial x^i}-\frac{\partial A_i}{\partial x^n}\tag{26}$$

where θ_0 is the charge density in the local inertial coordinate system in which the particle is at rest.

Since equation (26) must be the same for various points of the particle, the density ρ_0 must be proportional to the charge density θ_0: $\rho_0/\theta_0=m_0/q$, where q is the charge of the particle.

As $F_{in}=-F_{ni}$, we have the identity

$$g_{lk}\frac{dx^l}{ds}g^{ik}F_{in}\frac{dx^n}{ds}\equiv 0\tag{27}$$

Therefore, after multiplying equation (26) by $g_{lk}\,dx^l/ds$ we obtain the same equation (19) for the potential φ as we do in the case of $A_i=0$. From (19) and (26) we derive the following equation of the particle movement:

$$m_0^{(\varphi)}\left(c^2\frac{d^2x^k}{ds^2}+\frac{d\varphi}{ds}\frac{dx^k}{ds}-\frac{\partial\varphi}{\partial x_k}\right)-qg^{ik}F_{in}\frac{dx^n}{ds}=0\tag{28}$$

where q is the particle charge and $m_0^{(\varphi)}$ is its mass at rest when the nuclear potential equals φ. As a result of formulas (5) and (15), this mass can be determined by the formula

$$m_0^{(\varphi)}=m_0\exp(\varphi/c^2)\tag{29}$$

It must be noted that due to identity (27), after multiplying equation (28) by $g_{lk}\,dx^l/ds$, we get an identity which shows that equation (28) has no contradictions. The existence of this identity leads to the fact that the equation of the nuclear field (19) is the consequence of the differential law of energy-momentum conservation (1)–(2).

It must also be noted that the equation of particle movement (28), which has charge q and mass $m_0^{(\varphi)}$ at rest (29), can be derived from the classical action S:

$$\delta S=\delta\int_{P_0}^{P_1}\left(-m_0^{(\varphi)}c\,ds-\frac{q}{c}A_i\,dx^i\right)=0\tag{30}$$

where P_0 and P_1 are fixed space-time points, ds is the Minkowski interval, and δS is the variation of S.

To prove (30), we must calculate δS:

$$\delta S = -\int_{P_0}^{P_1}\left[m_0^{(\varphi)}c\,\frac{dx_i d(\delta x^i)}{ds} + \frac{dm_0^{(\varphi)}}{d\varphi}\frac{\partial\varphi}{\partial x^i}\delta x^i c\,ds + \frac{q}{c}A_i d(\delta x^i) + \frac{q}{c}\frac{\partial A_n}{\partial x^i}\delta x^i\,dx^n\right]$$

$$= \int_{P_0}^{P_1}\delta x^i\left[dm_0^{(\varphi)}c\,\frac{dx_i}{ds} + cm_0^{(\varphi)}d\left(\frac{dx_i}{ds}\right) - \frac{dm_0^{(\varphi)}}{d\varphi}\frac{\partial\varphi}{\partial x^i}c\,ds + \frac{q}{c}\left(\frac{\partial A_i}{\partial x^n} - \frac{\partial A_n}{\partial x^i}\right)dx^n\right] = 0 \quad (31)$$

From (31) and (29) we get the equation

$$m_0^{(\varphi)}\left(c^2\frac{d^2x_i}{ds^2} + \frac{d\varphi}{ds}\frac{dx_i}{ds} - \frac{\partial\varphi}{\partial x^i}\right) - qF_{in}\frac{dx^n}{ds} = 0 \quad (32)$$

which is identical to equation (28).

We turn now to nonlinear equation (19) describing the nuclear potential φ. First let us prove the theorem that in the stationary case $\partial\varphi/\partial x^0 = 0$ the solution φ of equation (19) is nonpositive: $\varphi \le 0$.

To prove this, let us assume that at some point M of three-dimensional space we have the inequality $\varphi(M) > 0$. Then, as $\varphi(\infty) = 0$, a certain point M_0 must exist at which the function φ has a positive maximum. Therefore, at the given point M_0 the following correlations must be fulfilled:

$$\varphi(M_0) > 0, \quad \partial\varphi(M_0)/\partial x^\alpha = 0, \quad \partial^2\varphi(M_0)/\partial x^{\alpha 2} \le 0, \quad \alpha = 1, 2, 3 \quad (33)$$

In the stationary case from (19) and (33) we obtain the following inequality:

$$0 > \sum_{\alpha=1}^{3}\frac{\partial^2\varphi}{\partial x^{\alpha 2}} - \frac{m_\pi^2 c^2}{\hbar^2}\varphi = \lambda\rho_0\exp\left(\frac{\varphi}{c^2}\right) \quad (34)$$

From (34) we get the inequality $\rho_0(M_0) < 0$, which is impossible. The given contradiction shows that the solution φ of equation (19), which is independent of the time factor and which vanishes at infinity, fulfills the inequality

$$\varphi \le 0 \quad (35)$$

It follows from (35) that in the stationary case when $\varphi \ne 0$ the mass $m_0^{(\varphi)}$ of a particle at rest is smaller than its value m_0 when $\varphi = 0$. This property demonstrates the well-known fact that the nucleus bound energy is negative.

Let us consider equation (19) in the stationary spherically symmetric case. Then equation (19) takes the form

$$\varphi'' + 2\varphi'/r - \nu^2\varphi = \lambda\rho$$

$$\varphi = \varphi(r), \quad \rho = \rho(r), \quad \nu = m_\pi c/\hbar, \quad r = [(x^1)^2 + (x^2)^2 + (x^3)^2]^{1/2} \quad (36)$$

As is known, we can transform the differential equation (36) to an integral equation describing the function $\varphi(r)$ which fulfills the condition $\varphi(\infty) = 0$. This integral equation has the form

$$\varphi(r) = \frac{\lambda}{2\nu r} \int_0^\infty z\rho(z)[\exp(-\nu(r+z)) - \exp(-\nu|r-z|)]\,dz \quad (37)$$

By performing the differentiation of equation (37) and then the integration by parts of the right-hand side of the equation we obtain

$$\varphi'(r) = -\frac{\lambda}{2\nu^3 r^2} \int_0^\infty \rho'(z) M(r, z)\,dz \quad (38)$$

where

$$\begin{aligned} M(r, z) &= K(r, -z) - K(r, z) \\ K(r, z) &= \exp(-\nu|r-z|)(1 - \nu^2 rz + \nu|r-z|) \end{aligned} \quad (39)$$

Note the following property of the function $M(r, z)$. It is not difficult to prove that the function $M(r, z)$ is nonnegative:

$$M(r, z) \geq 0 \quad (40)$$

After substituting (17) for ρ in equation (38) we obtain

$$\varphi'(r) = -\frac{\lambda}{2\nu^3 r^2} \int_0^\infty \exp\left(\frac{\varphi(z)}{c^2}\right) M(r, z)\left[\rho_0'(z) + \rho_0(z)\frac{\varphi'(z)}{c^2}\right] dz \quad (41)$$

It is necessary to note that when equation (26) was obtained attention was drawn to the fact that the density ρ_0 was proportional to the charge density θ_0. As is known, the charge density θ_0 of the nuclei decreases when the radius r increases (Naumov, 1984). Hence we have the inequality $\rho'(r) \leq 0$. Therefore, if the second summand had been zero in the square brackets in equation (41) we would have the inequality $\varphi'(r) \geq 0$ and as a result the nuclear forces would attract.

But it is well known that nuclear forces can also repulse (Naumov, 1984). This property can be explained by the second summand in the square brackets in equation (41). This summand appears in equation (41) because of formula (17).

When $r \gg 1/\nu$ we have the Yukawa solution for which the derivative $\varphi'(r)$ is nonnegative: $\varphi'(r) \geq 0$. To find a condition which must be fulfilled to have the negative derivative $\varphi'(r)$ at certain points r of the region $0 < r \leq r_0 \sim 1/v$, we use the following functions:

$$\begin{aligned} I_1(r) &= -\frac{\lambda}{2v^3r^2}\int_0^\infty \exp\left(\frac{\varphi(z)}{c^2}\right) M(r, z)\rho_0'(z)\, dz \\ I_2(r) &= \frac{\lambda}{2v^3r^2c^2}\int_0^\infty \exp\left(\frac{\varphi(z)}{c^2}\right) M(r, z)\rho_0(z)|\varphi'(z)|\, dz \end{aligned} \tag{42}$$

It is seen that the function $I_2(r)$ is always positive. Let us assume that at a certain point $r = \bar{r}$ we have the following inequality:

$$I_2(\bar{r}) > I_1(\bar{r}) \tag{43}$$

Then we can prove the statement that a certain point $r = r_1$ exists at which the derivative $\varphi'(r_1)$ is negative: $\varphi'(r_1) < 0$.

To prove this, let us assume that $\varphi'(r) \geq 0$ at any point r. Then from (41)–(43) we find

$$\varphi'(r) = I_1(\bar{r}) - I_2(\bar{r}) < 0$$

Therefore, we get a contradiction. So, if we have a certain point $\bar{r}$ at which inequality (43) is fulfilled, then a region of the point r must exist at which $\varphi'(r) < 0$ and hence in this case the nuclear forces repulse.

It follows from condition (42)–(43) that there are two cases in which the inequality $\varphi' < 0$ can hold:

1. The case of nuclear saturation. This case can take place if the function $|\varphi(r)|$ reaches a sufficiently large value at the point r in the region $0 \leq r \leq r_0 \sim 1/v$: $|\varphi| \gtrsim c^2$.

2. The case of a repulsive nuclear core. This case can take place if the function $|\varphi'(r)|$ reaches a sufficiently large value at the point r in the region $0 < r \leq r_0 \lesssim 1/v$: $|\varphi'| \gtrsim c^2|\rho_0'|/\rho_0$.

Therefore, the nonlinear equation (19) describing the nuclear potential φ allows the interpretation of the experiments (Naumov, 1984) in which nuclear forces repulse as well as attract.

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